

# Parametrically driven dark solitons

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We show that unlike the bright solitons, the parametrically driven kinks are immune from instabilities for all dampings and forcing amplitudes; they can also form stable bound states. In the undamped case, the two types of stable kinks and their complexes can travel with nonzero velocities.

The parametric driving is well known to be an efficient way of compensating dissipative losses of solitons in various media. Examples include surface solitons in vertically oscillating layers of water [1, 2]; light pulses in optical fibers under phase-sensitive amplification [3] and in Kerr-type optical parametric oscillators [4]; magnetisation solitons in easy-plane ferromagnets exposed to oscillatory magnetic fields in the easy plane [5]. A serious problem associated with the parametric energy pumping, however, is that the driven solitons are prone to oscillatory instabilities which set in as the driver's strength exceeds a certain — often rather low — threshold [5, 6].

With a few notable exceptions, the parametrically driven solitons considered so far had the form of pulses decaying to zero at spatial infinities. These were solutions of the nonlinear Schrödinger (NLS) equation with the “self-focusing” nonlinearity:

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi - \psi = h\psi^* - i\gamma\psi, \quad (1)$$

where  $\psi$  is the amplitude of a nearly-harmonic stationary wave oscillating with half the driving frequency,  $\gamma$  the damping coefficient,  $h$  the driving strength, and  $^*$  indicates complex conjugation. However, in a number of applications the amplitude equation of the parametrically driven wave turns out to have the nonlinearity of the “defocusing” type:

$$i\psi_t + \frac{1}{2}\psi_{xx} - |\psi|^2\psi + \psi = h\psi^* - i\gamma\psi. \quad (2)$$

In fluid dynamics, the “defocusing” parametrically driven NLS (2) describes the amplitude of the oscillation of the water surface in a vibrated channel with a large width-to-depth ratio [2, 7]. (On the contrary, the “focusing” equation (1) pertains to the case of narrow channels.) The same equation (2) arises as an amplitude equation for the upper cutoff mode in the parametrically driven damped nonlinear lattices [8]. In the optical context, it was derived for the doubly resonant  $\chi^{(2)}$  optical parametric oscillator in the limit of large second-harmonic detuning [9]. Next, in the absence of damping, stationary solutions  $\psi = M_y + iM_z$  of eq.(2) minimise the Ginzburg-Landau free energy for the anisotropic  $XY$  model,  $F = \int \mathcal{F} dx$ , where

$$\mathcal{F} = \frac{1}{2}(\partial_x \mathbf{M})^2 - (1+h)\mathbf{M}^2 + \frac{1}{2}\mathbf{M}^4 + 2hM_y^2 + \mathcal{F}_0,$$

and  $\mathbf{M} = (0, M_y, M_z)$ . This model was used to study domain walls in easy-axis ferromagnets near the Curie

point [10]. Nonstationary magnetisation configurations were considered in the overdamped limit:  $\psi_t = -\delta F/\delta\psi^*$  [11, 12]. The damped hamiltonian dynamics  $\psi_t = -i\delta F/\delta\psi^* - \gamma\psi$  provides a sensible alternative; this is precisely our eq.(2). Finally, for  $\gamma = 0$  there is yet another, independent, magnetic interpretation of eq.(2); this will be introduced below.

Localised structures characteristic of a defocusing medium are domain walls, or kinks, also known as “dark solitons” in the context of nonlinear optics. The purpose of this note is to explore the stability and bifurcations of the parametrically driven kinks and their bound states.

Two stationary kink solutions of (2) are available in literature. One is usually called the Néel, or Ising, wall:

$$\psi_N(x) = iA \tanh(Ax)e^{-i\theta}. \quad (3)$$

Here  $A^2 = 1 + \sqrt{h^2 - \gamma^2}$  and  $\theta = -\frac{1}{2} \arcsin \frac{\gamma}{h}$  [2, 7, 9]. For  $\gamma = 0$ , the Néel wall coexists with the Bloch wall:

$$\psi_B(x) = -iA \tanh(2\sqrt{h}x) \pm \sqrt{1 - 3h} \operatorname{sech}(2\sqrt{h}x), \quad (4)$$

$A^2 = 1 + h$  [13]. Originally, the “magnetic” terminology was motivated merely by the fact that  $|\psi| = 0$  in the core of the wall (3), and so the point  $x = 0$  is a phase defect similar to Néel points in solid state physics [12]. On the contrary, in the core of the Bloch wall the phase changes smoothly — and this is analogous to Bloch walls in ferromagnets. Below we show that the analogy with magnetism is in fact much deeper than originally thought. Letting  $h = \gamma = 0$ , the Néel wall becomes the usual, undriven, dark soliton whereas the Bloch wall degenerates into a flat solution.

Both Bloch and Néel walls admit a clear interpretation in other physical contexts as well. For example, when eq.(3) is used to model the Faraday resonance in water [2, 7] or chains of coupled pendula [8], both solutions describe transitions between two domains oscillating  $180^\circ$  out of phase. The phase of the oscillation is discontinuous across the Néel wall and the amplitude changes over a narrow region; hence the wall appears as a highly localised defect. Conversely, the Bloch wall has a smooth helicoidal structure, with the amplitude varying over a wider interval. It might therefore be tempting to expect that in the region of their coexistence ( $h < \frac{1}{3}$ ), the Néel wall should be unstable against the decay into the “smoother” (Bloch) wall, and this is indeed the case in the Ginzburg-Landau and Klein-Gordon counterparts of

eq.(2) [11, 14, 15]. Surprisingly, the NLS dynamics turn out to be very different.

We will show, numerically, that when  $\gamma = 0$ , both walls can stably move with nonzero velocities and form stable stationary and oscillatory, quiescent and travelling bound states. The resulting bifurcation diagram will then be interpreted using two integrals of motion of eq.(2), and a relation of our model to biaxial ferromagnets established. Turning to the dissipative case, we will give an analytical proof of the stability of the Néel wall for all  $h$  and  $\gamma$  and describe stable bound states formed by the damped walls.  $\square$

Both static kinks, (3) and (4), belong to a broader class of uniformly moving solutions of the form  $\psi(x - Vt)$ . We found these by solving equation  $\frac{1}{2}\psi_{xx} - iV\psi_x - |\psi|^2\psi + \psi = h\psi^*$  [18]. Fig.1(a) shows the momentum of the travelling wall as a function of its velocity. The momentum  $P = \text{Im} \int \psi_x \psi^* dx$  is one of the two conserved quantities of eq.(3) with  $\gamma = 0$ , and hence is a natural choice for the integral characteristic of its solutions. The second integral is energy, and it will also be used below:

$$E = \text{Re} \int \left( \frac{|\psi_x|^2}{2} + \frac{|\psi|^4}{2} - |\psi|^2 + h\psi^2 + \frac{A^4}{2} \right) dx. \quad (5)$$

(Here  $A^2 = 1 + h$ .) The stability of the travelling walls was examined [18] by computing eigenvalues  $\lambda$  of

$$\mathcal{H}\vec{\varphi} = \lambda J\vec{\varphi}, \quad (6)$$

where the column  $\vec{\varphi} = (u, v)^T$ , the operator  $\mathcal{H}$  is given by

$$\mathcal{H} = -\frac{I}{2}\partial_x^2 + \begin{pmatrix} 3\mathcal{R}^2 + \mathcal{I}^2 + h & 2\mathcal{R}\mathcal{I} - V\partial_x + \gamma \\ 2\mathcal{R}\mathcal{I} + V\partial_x - \gamma & \mathcal{R}^2 + 3\mathcal{I}^2 - h \end{pmatrix},$$

and  $J$  is the antisymmetric matrix with  $J_{21} = -J_{12} = 1$ . Eq.(6) is obtained by linearising eq.(2) about  $\psi = \mathcal{R} + i\mathcal{I}$  in the co-moving frame, and letting  $\delta\psi = (u + iv)e^{\lambda t}$ .

In a striking contrast to the diffusive and relativistic dynamics [11, 14, 15], our numerical analysis of eq.(6) reveals that not only the stationary Néel wall, but the entire branch of travelling kinks in Fig.1(a) is stable. This multistability admits a simple explanation in terms of the energy and momentum, though. The energy of the stationary Néel wall,  $E_N = \frac{4}{3}(1+h)^{3/2}$ , is greater than that of the stationary Bloch wall,  $E_B = 4\sqrt{h} - \frac{4}{3}h^{3/2}$ , and so one might expect  $\psi_N$  to decay into  $\psi_B$  plus radiation waves — as in the relativistic case [14]. However, unlike their relativistic counterparts, our Bloch and Néel walls have unequal momenta, with  $P_B > P_N$  (see Fig.1(a)) — and this makes the  $\psi_N \rightarrow \psi_B$  decay impossible.

Next, our simulations of the time-dependent eq.(2) show that two stationary Bloch walls with opposite chiralities (i.e. opposite signs in (4)) can attract and form a motionless breather-like bound state (Fig.1(b)). An attraction of stationary Bloch and Néel walls results in a

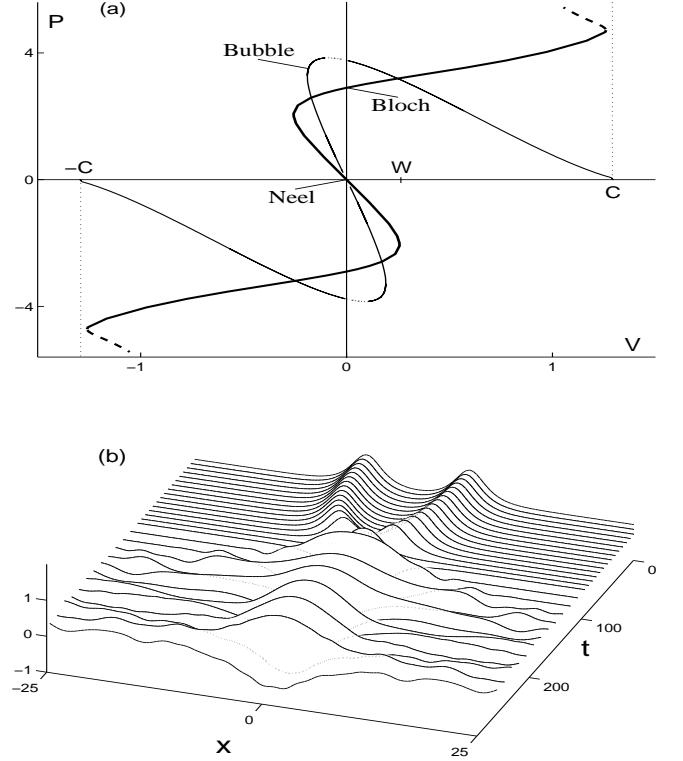


FIG. 1: (a) The momentum of the travelling Bloch and Néel walls (thick) and their nonoscillatory bubble-like complex (thin line). For  $|V|$  close to  $c$ , the wall attaches a small-amplitude bubble on each flank; this accounts for the turn of the thick curve near  $|V| = c$ . The dotted segments of the continuous branches indicate unstable solutions. (b) The formation of an oscillatory breather-like complex of two walls. (Only the real part of  $\psi$  is shown for visual clarity.) In (a),  $h = \frac{1}{15}$ ; in (b),  $h = 0.1$ .

moving breather. There also exist nonoscillatory, bubble-like, bound states of  $\psi_N$  and  $\psi_B$ . Note that unlike their parent walls, all of these complexes approach the same background as  $x \rightarrow \infty$  and  $x \rightarrow -\infty$ . The bubble-like solitons admit most transparent physical interpretation: they describe “islands” of one stable phase in the sea of the other one, e.g. patches oscillating 180° out of phase with the rest of the vibrated water channel or chain of pendula. Below we focus on the bubbles and relegate the breathers to a separate publication.

For each  $h$  there is a one-parameter family of motionless bubbles, the parameter being the separation distance  $z$  between the two walls. (Accordingly, there are two zero eigenvalues in the spectrum of the operator (6) associated with each bubble, one translational and the other one corresponding to variations in  $z$ ). There is a particular separation  $z = \zeta$  for which the bubble is symmetric:  $\psi_\zeta^*(-x) = -\psi_\zeta(x)$ . The symmetric bubble turns out to have the largest momentum over bubbles with various  $z$ , while  $\zeta$  is the smallest possible separation:  $z \geq \zeta$ . More importantly, it is the only *stable* bubble. All nonsymmetric bubbles ( $z > \zeta$ ) were found to have a pair of nonzero

real eigenvalues  $\pm\lambda$  in their spectrum. As  $z \rightarrow \zeta$ , the pair converges at the origin and so the symmetric bubble has *four* zero eigenvalues.

Let  $\pm\lambda$  be a pair of eigenvalues diverging from zero as  $z$  grows from  $\zeta$  and assume that  $\lambda = \epsilon^{1/2}\lambda_1 + \dots$  for small  $\epsilon = z - \zeta$ . Then the associated eigenfunction  $\vec{\varphi}$  expands as  $\vec{\varphi} = \vec{\varphi}_0 + \epsilon^{1/2}\vec{\varphi}_1 + \dots$ , where  $\vec{\varphi}_0$  is a linear combination of the two zero modes:  $\vec{\varphi}_0 = (C_1\partial_x\vec{\psi} + C_2\partial_z\vec{\psi})|_{z=\zeta}$ , with  $\vec{\psi} \equiv (\mathcal{R}, \mathcal{I})^T$ . (The other two zero eigenvalues have only *generalised* eigenvectors associated with them.) Substituting this into (6) and using  $\mathcal{H}_z = \mathcal{H}_\zeta + \epsilon\mathcal{H}_1 + \dots$ , the order  $\epsilon^{1/2}$  yields  $\mathcal{H}_\zeta\vec{\varphi}_1 = \lambda_1 J\vec{\varphi}_0$ . A bounded  $\vec{\varphi}_1(x)$  exists only if  $\lambda_1 J\vec{\varphi}_0$  is orthogonal both to  $\partial_z\vec{\psi}$  and  $\partial_x\vec{\psi}$ . This orthogonality condition amounts to  $(dP/dz)|_{z=\zeta} = 0$ , and the latter relation explains why the momentum has to reach its maximum at the value of  $z$  for which a pair of real eigenvalues converges at the origin.

The other implication of the relation  $dP/dz = 0$  is that it allows the symmetric bubble to be continued to  $V \neq 0$  [19]. The resulting branch of moving bubbles is shown in Fig.1(a). As  $|V| \rightarrow c = (1 + 2h + \sqrt{4h(1+h)})^{1/2}$ , which is the minimum phase velocity of linear waves, the bubble degenerates into the flat background, whereas when  $V, P \rightarrow 0$ , it transforms into a pair of Néel walls with the separation  $z \rightarrow \infty$ . The entire branch of moving bubbles is stable, with the exception of a small region between  $V = 0$  and the point of the maximum  $|P|$  inside which a real pair  $\pm\lambda$  occurs (Fig.1(a)). The change of stability at points where  $dP/dV = 0$ , is explained in [19].

The diagram Fig.1(a) can further justify referring to the kinks (3) and (4) as Néel and Bloch walls. In fact stationary nonchiral interfaces called Néel walls are known in uniaxial ferromagnets, where they coexist with chiral (Bloch) walls. When the axial symmetry is broken, the two types of walls can move; this occurs in particular in easy-axis ferromagnets with the second, weaker, anisotropy axis ( $\beta, \epsilon < 0$  and  $H = 0$  in eq.(8) below) [16]. The  $P(V)$  curve for the easy-axis walls is qualitatively similar to our Fig.1(a): The Néel wall's momentum  $P_N = 0$  while  $P_B \neq 0$ ; as the velocity  $V$  grows, the two branches are drawn closer together and finally merge. The limit velocity  $V = w$  is known as the Walker's velocity [16]. The  $E(V)$  curves are also similar.

This analogy suggests that there could be a link between the time-dependent NLS (3) and Heisenberg ferromagnets and indeed, there is one. Consider a quasi-one-dimensional ferromagnet with a weakly anisotropic easy plane  $(M_x, M_y)$ , in the external stationary magnetic field along  $M_z$ . The magnetisation vector  $\mathbf{M} = (M_x, M_y, M_z)$  lies on the sphere,  $\mathbf{M}^2 = M_0^2$ , and satisfies the (damped) Landau-Lifshitz equation [16]:

$$\frac{\hbar}{2\mu_0} \mathbf{M}_\tau = \mathbf{M} \times \frac{\delta}{\delta \mathbf{M}} \int \mathcal{W} d\xi - \lambda \mathbf{M} \times \mathbf{M}_\tau, \quad (7)$$

$$\mathcal{W} = \frac{\alpha}{2}(\partial_\xi \mathbf{M})^2 + \frac{\beta}{2}M_z^2 + \frac{\epsilon\beta}{2}M_x^2 - HM_z + \mathcal{W}_0, \quad (8)$$

with  $\beta > 0$ . If the anisotropy parameter  $\epsilon$  is small and the field  $H$  is close to  $\beta M_0$ :  $H = \beta M_0 - \epsilon q$ , the vector  $\mathbf{M}$  will stay near the northern pole of the sphere. Choosing  $s \equiv qM_0 - \beta M_0^2/2 > 0$  for  $\epsilon > 0$  and  $s < 0$  for  $\epsilon < 0$ , we define  $M_x + iM_y = (2\epsilon s/\beta)^{1/2}\psi^*$ . Assuming that the relaxation constant  $\lambda$  is  $\mathcal{O}(\epsilon^{1/2})$  or smaller, and that  $\mathbf{M}$  depends only on “slow” variables  $x = (\epsilon s/2\alpha M_0^2)^{1/2}\xi$  and  $t = (2\epsilon\mu_0 s/\hbar M_0)\tau$ , eq.(7)-(8) reduces to eq.(2) with  $h = \beta M_0^2/(2s)$  and  $\gamma = 0$ . Note that the resulting NLS is undamped — although the original Landau-Lifshitz equation did include a small damping term. The effect of damping will become noticeable only on time scales longer than  $\epsilon^{-1}$ ; these are not captured by eq.(2). Also note that despite the analogy between the easy-axis and easy-plane ferromagnets, there are important physical differences. In particular in the easy-axis case the walls interpolate between  $\mathbf{M}/M_0 = \pm(0, 0, 1)$  while in our case they separate domains with  $\mathbf{M} \sim (0, \pm|\epsilon|^{1/2}, 1)$ .  $\square$

Proceeding to the damped situation,  $\gamma \neq 0$ , our first goal is to demonstrate the stability of the Néel wall. We let  $\psi(x, t) = \psi_N(x) + \delta\psi(x, t)$ , where

$$\delta\psi(x, t) = [u(X) + iv(X)]e^{(\mu-\Gamma)t-i\theta}; \quad (9)$$

$X = Ax$ ,  $T = A^2t$ ,  $\Gamma = A^{-2}\gamma$  and  $\mu$  is complex. Linearising eq.(2) in small  $\delta\psi$  we obtain an eigenvalue problem

$$(L_0 + \epsilon)v = (\mu - \Gamma)u, \quad L_1 u = -(\mu + \Gamma)v, \quad (10)$$

where  $\epsilon = 2 - 2/A^2$  and  $L_0$  and  $L_1$  are the Schrödinger operators with familiar spectral properties:

$$L_0 \equiv -\frac{1}{2}\partial_X^2 - \operatorname{sech}^2 X, \quad L_1 \equiv L_0 + 2\tanh^2 X.$$

Introducing  $\nu^2 = \mu^2 - \Gamma^2$  and  $w = \nu^{-1}(\mu + \Gamma)v$  [5], we eliminate  $\Gamma$  from the eigenvalue problem (10):

$$(L_0 + \epsilon)w = \nu u, \quad L_1 u = -\nu w. \quad (11)$$

Now we will show that  $\nu^2 < 0$  for all  $0 \leq \epsilon < 2$ , so that  $\mu^2 < \Gamma^2$  and all perturbations decay to zero as  $t \rightarrow \infty$ .

The operator  $L_1$  has a zero eigenvalue, with the associated eigenfunction  $y_0(X) = \operatorname{sech}^2 X$ , and no negative eigenvalues. Consequently, on the subspace  $\mathcal{R}$  defined by

$$\int u(X)y_0(X)dX = 0, \quad (12)$$

there exists an inverse operator  $L_1^{-1}$  and so (11) becomes  $(L_0 + \epsilon)u = -\nu^2 L_1^{-1}u$ , with  $L_0 + \epsilon$  symmetric and  $L_1^{-1}$  a positive operator. The smallest eigenvalue  $-\nu_0^2$  is given by the minimum of the Rayleigh quotient:

$$-\nu_0^2 = \min_{u \in \mathcal{R}} \frac{\int u(L_0 + \epsilon)udX}{\int uL_1^{-1}udX}. \quad (13)$$

To prove that  $-\nu_0^2 > 0$  it is sufficient to show that the minimum of the quadratic form  $\int u(L_0 + \epsilon)udX$  is positive on  $\mathcal{R}$  [17]. Assuming that  $u(X)$  are normalised by

$\int u^2 dX = 1$ , the minimum is attained on the solution  $u(X)$  to the nonhomogeneous boundary-value problem

$$(L_0 + \epsilon)u(X) = \eta u(X) + \alpha y_0(X), \quad (14)$$

where  $\eta$  and  $\alpha$  are the Lagrange multipliers. The minimum equals  $\eta$  — provided  $\eta$  and  $\alpha$  are chosen so that the  $u(X)$  satisfies eq.(12) and the normalization constraint.

The operator  $L_0$  has a single discrete eigenvalue  $E_0 = -\frac{1}{2}$  with the eigenfunction  $z_0(X) = (1/\sqrt{2})\text{sech}X$ , and the continuous spectrum of eigenvalues  $E(k) = k^2$ , with

$$z_k(X) = \frac{ik + \tanh X}{ik - 1} e^{-ikx}, \quad -\infty < k < \infty. \quad (15)$$

Expanding  $y_0$  and  $u$  over the complete set  $\{z_0; z_k\}$  gives

$$\begin{aligned} y_0(X) &= Y_0 z_0(X) + \int Y(k) z_k(X) dk, \\ u(X) &= U_0 z_0(X) + \int U(k) z_k(X) dk. \end{aligned}$$

Substituting into (14) and using the orthogonality of the functions in the set produces  $U(k) = \alpha(k^2 + \epsilon - \eta)^{-1}Y(k)$  and  $U_0 = \alpha(E_0 + \epsilon - \eta)^{-1}Y_0$ . Using these in (12) gives

$$g_\epsilon(\eta) \equiv \frac{Y_0^2}{E_0 + \epsilon - \eta} + \int_{-\infty}^{\infty} \frac{|Y(k)|^2}{k^2 + \epsilon - \eta} dk = 0. \quad (16)$$

The minimum of the quadratic form  $\int u(L_0 + \epsilon)udX$  is given by the smallest root  $\eta^*$  of the function (16). The function  $g_\epsilon(\eta)$  is increasing for  $-\infty < \eta \leq \epsilon$ , apart from the point  $\eta = E_0 + \epsilon$  where it drops from  $+\infty$  to  $-\infty$ . As  $\eta \rightarrow -\infty$ ,  $g_\epsilon(\eta) \rightarrow +0$ ; as  $\eta \rightarrow \epsilon$ ,  $g_\epsilon(\eta)$  tends to a finite value. (This follows from the fact that

$$Y(k) = \frac{ik}{1 + ik} \frac{\pi k/2}{\sinh(\pi k/2)},$$

hence the integral in (16) converges for all  $\eta \leq \epsilon$ .) Consequently, there is only one root  $\eta^*$  and its sign is opposite to the sign of  $g_\epsilon(0)$ . Since  $\partial g_\epsilon(\eta)/\partial \epsilon < 0$ , we have  $g_\epsilon(0) < g_0(0)$  while the value  $g_0(0)$  can be calculated as

$$g_0(0) = \frac{Y_0^2}{E_0} + \int_{-\infty}^{\infty} \frac{|Y(k)|^2}{k^2} dk = \int y_0 L_0^{-1} y_0 dX. \quad (17)$$

Noticing that  $L_0^{-1} y_0(X) = -1 + c \tanh X$ , with  $c$  an arbitrary constant, eq.(17) yields  $g_0(0) = -2$  and hence  $\eta^*$  cannot be negative for any  $\epsilon$ . Thus  $-\nu^2 > 0$  and the Néel wall is stable for all  $h$  and  $\gamma$  (with  $h \geq \gamma \geq 0$ ).

Are there any other attractors for nonzero  $\gamma$ ? When  $\gamma \neq 0$ , the momentum is, in general, changing with time:  $\dot{P} = -2\gamma P$ , and therefore a uniformly moving soliton has to satisfy  $P = 0$ . Since both curves in Fig.1(a) cross the  $P = 0$  axis only at the stationary Néel walls, we conclude that no other solutions persist for small nonzero  $\gamma$ . This does not, however, exclude the existence of new

solutions for *larger*  $\gamma$ . Our numerical analysis has, in fact, revealed a window of  $h$  values,  $\gamma < h \lesssim 0.35 + 0.8\gamma$  (with  $0.1 \leq \gamma \leq 0.85$ ) where two Néel walls attract and form a stable stationary bubble.  $\square$

In conclusion, the remarkable stability of the damped-driven kinks and their bound states is in sharp contrast with stability properties of the bright solitons. The stable coexistence of two types of domain walls and their complexes in the undamped case is also worth emphasizing. This multistability is not observed in the parametrically driven Klein-Gordon and Ginzburg-Landau equations and is due to the availability of the momentum integral which takes different values on different solutions.

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